MOSCOW MATHEMATICAL JOURNAL
Volume 1, Number 3, July–September 2001, Pages 407–419

STEADY SOLUTIONS FOR FIFO NETWORKS

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Dedicated to Robert Minlos on the occasion of his 70th birthday

Abstract. We consider the fluid model of a reentrant line with FIFO discipline and look for solutions with constant flows (steady solutions). In the case of constant viscosities we prove the uniqueness of such a solution. If viscosities are different, we present an example with multiple steady solutions. We also prove that for some classes of reentrant lines uniqueness holds even if the viscosities are different.

2000 Math. Subj. Class. 90B10, 94C99, 37–XX.

Key words and phrases. Kelly networks, fluid models, uniqueness of steady solution, fixed points.

1. Introduction

In this paper we study multi-class networks with the FIFO serving discipline. We consider deterministic fluid models and look for solutions with stationary flows (steady solutions). The class of such solutions is the same for some other disciplines in multi-class networks like LIFO or processor sharing.

Fluid limits and fluid models for queueing networks were first introduced in [9]. A similar scaling for stochastic processes of different kind (random walks in finite-dimensional positive orthants) was studied by V. A. Malyshev and his coauthors, see references in the review [7]. Analogous deterministic discrete models were also studied in [6].

Trajectories of a fluid model are solutions of a complicated system of functional equations with delay in the space of monotone Lipschitz continuous vector-functions (see [4, 10]). This system of equations can be derived from the Euler limit for the stochastic process which describes the evolution of a queueing network in time. The whole construction allows one to prove the convergence of stochastic processes to the limiting fluid dynamics. Note that the system of fluid equations also depends on "excessive" variables (vanishing in the Euler scaling). These variables provide

Received July 31, 2001; in revised form September 11, 2001.

Third named author supported in part by RFBR Grant 99-01-00003, CRDF Grant RM-2085, and INTAS Grant.

Fourth named author supported in part by RFBR Grants 00-01-00571 and 00-15-96116, and INTAS Grant.

the same phase space for both stochastic and limiting deterministic dynamics (see, for instance, [8]).

In [9] the fluid dynamics formalism was developed in order to obtain ergodicity conditions for original stochastic processes in terms of the behavior of the limiting fluid trajectories. The fluid behavior was found to be highly nontrivial. Surprisingly, the ergodicity conditions happened to be quite different from naive "common sense" expectations that the system is ergodic whenever the workload at each node is less than 1 (see [6], [9] for the first counterexamples and [1, 2] for those in the FIFO context). This phenomenon is closely related to the nonuniqueness of trajectories for fluid dynamics and self-similarity of fluid trajectories (see [10, 8]). In examples mentioned above, it follows from self-similarity that there exist infinitely many nontrivial solutions with zero initial conditions (the empty state of the queueing system). Moreover, for all these fluid trajectories the set of non-empty nodes is switched infinitely often in any finite neighbourhood of $t = 0$. Below we refer to such complicated behavior as turbulent-like. Let us also mention the study by V. A. Malyshev [7] of similar bifurcation phenomena for limiting trajectories in the Euler scaling in the case of random walks in finite-dimensional positive orthants.

To the best of our knowledge, the question of uniqueness of a solution is open for all nontrivial fluid models with infinite-dimensional state space (like the space of Lipschitz functions in the FIFO case, in contrast to networks with priority disciplines and Jackson networks for which the phase space is finite-dimensional). In this paper we consider the simplest fluid trajectories with constant flow rates for all types of fluid originating from the empty state of the network (*steady solutions*).

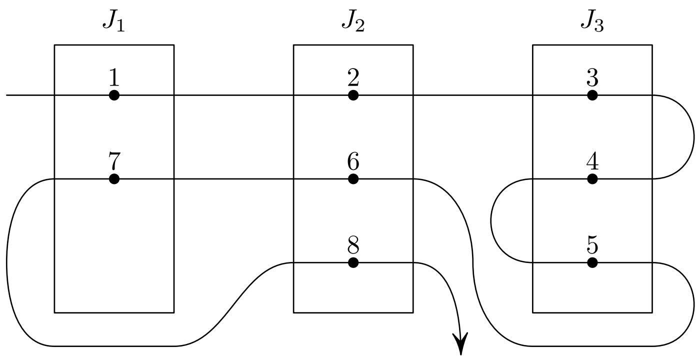

FIGURE 1. An example of reentrant line

Let us begin with a single reentrant line with K nodes. Such a line is characterized by a partition of a finite set $\{1, \ldots, n\}$ of fluid classes into K disjoint subsets $J_i = \{j^i_1, \ldots, j^i_{n_i}\}$ (nodes), see Fig. 1. Denote by $i(j)$ the index i such that $j \in J_i$. The maximal service capacities of nodes are given implicitely by the vector $(v_1, \ldots, v_n)$ of positive numbers (mean serving times or *viscosities*). The prelimiting random process describes a queueing network consisting of K FIFO queues.

Customer arrivals at the network are described by the stationary metrically transitive flow with average interarrival times equal to 1. Each customer is served in n stages. On the jth stage the customer joins queue $i(j)$ and its mean service time (the mean time during which the customer is the first in the queue) is equal to v_j. We say that this customer is of class j until it is served at the station $i(j)$ and joins the next queue $i(j)$ (which may, incidentally, be the same queue) or leaves the network forever if $j = n$. We say that the reentrant line is of *Kelly* type if the mean service time at node i does not depend on $j \in J(i)$, i.e., $v_j = V_i$ for all $j \in J_i$, $i = 1, \ldots, K$.

A steady solution is described by an $(n+1)$-vector $x = (x_0, x_1, \ldots, x_n)$, where x_j, $j = 1, \ldots, n-1$ is the rate of flow leaving class j and joining class $j+1$ or leaving the network forever if $j = n$. The vector x must satisfy the conditions

$$\begin{aligned} &(1) \quad x_0 = 1, \\ &(2) \quad \textstyle\sum_{j \in J_i} x_j v_j \le 1, \quad i = 1, \ldots, K. \\ &(3) \quad x_s \max\bigl\{1, \textstyle\sum_{j \in J_i} x_{j-1} v_j\bigr\} = x_{s-1} \text{ for all } s \in J_i, \quad i = 1, \ldots, K. \end{aligned} \tag{1.1}$$

The value of x_0 is the exogenous inflow. Condition (2) gives the service capacity restriction on the outcoming flow from node i. Condition (3) says that an equal fraction of each incoming flow to any class of node i passes through to the next class. This is a result of the FIFO discipline at the node and of the steadiness of the solution.

Notice that the same condition also holds in the case of some other disciplines in multi-class networks, like LIFO or processor sharing. Conditions (1.1) are derived from the more general ones

$$\begin{aligned} &(\tilde{1}) \quad \tilde{x}_0 > 0 \text{ arbitrary}, \\ &(\tilde{2}) \quad \textstyle\sum_{j \in J_i} \tilde{x}_j \tilde{v}_j \le c_i, \quad i = 1, \ldots, K. \\ &(\tilde{3}) \quad \tilde{x}_s \max\bigl\{c_i, \textstyle\sum_{j \in J_i} \tilde{x}_{j-1} \tilde{v}_j\bigr\} = \tilde{x}_{s-1} \text{ for all } s \in J_i, \quad i = 1, \ldots, K. \end{aligned}$$

by the rescaling $x_j = \tilde{x}_j / \tilde{x}_0$ and $v_j = \tilde{v}_j \tilde{x}_0 / c_{i(j)}$.

Note that, if

$$\sum_{j \in J_i} v_j < 1, \tag{1.2}$$

for any node J_i, $i = 1, \ldots, K$, then there exists a unique trivial steady solution $x_j \equiv 1$ for $j = 1, \ldots, n$. It is also known (see [3]) that if (1.2) holds, then, for a reentrant line of Kelly type and for any initial state of fluid, the total amount of fluid in the network vanishes in finite time.

The main theorem of this paper asserts that in the case of reentrant lines of Kelly type there exists a unique steady solution for any steady input without restriction (1.2). Let us notice that for solutions that are not necessary steady the question of uniqueness remains open. We conjecture that for reentrant lines of Kelly type the uniqueness of fluid trajectories persists in the general nonsteady case. On the contrary, in the case of different viscosities and more than one node, it is possible to have multiple steady solutions as Example 2.6 demonstrates.

As we have mentioned above, there might exist many nonstationary trajectories originating at the same initial state of the network. However, the nonuniqueness of steady solutions is a new and unexpected phenomenon. This fact implies the existence of even more complicated singularities of fluid trajectories than the turbulent-like examples mentioned above.

In Section 2 we introduce and study special dynamical systems (mappings) that arise naturally from the model presented above. Fixed points of these mappings correspond to steady solutions of the corresponding reentrant lines. We present an example of nonuniqueness of a steady state for a general reentrant line and prove other particular results for general reentrant lines. In Section 3 the uniqueness theorem for networks of Kelly type is proved (Main Theorem).

2. Dynamical systems

Let us define a mapping $M\colon \mathbb{R}_+^n \to \mathbb{R}_+^n$ by the following rule: $M\colon (x_1, \dots, x_n) \to (\tilde{x}_1, \dots, \tilde{x}_n)$, where

$$\tilde{x}_s = \frac{x_{s-1}}{\max\{1, \sum_{j\in J_i} x_{j-1}v_j\}} \quad \text{for} \quad s \in J_i, \quad i = 1, \dots, K,$$

and $x_0 = 1$. Clearly $x \in \mathbb{R}_+^n$ satisfies (1.1) iff $Mx = x$. Hence, the question of existence and uniqueness of a steady solution (1.1) can be reduced to the analysis of fixed points of the mapping M.

The image $M\mathbb{R}_+^n$ belongs to the compact convex set

$$X = \Big\{x \in \mathbb{R}_+^n\colon \sum_{j\in J_i} v_j x_j \le 1, \ \ i = 1, \dots, K\Big\}.$$

Since M is continuous, the Brouwer principle immediately implies the following lemma.

Lemma 2.1. *There exists an $x \in X \subset \mathbb{R}_+^n$ such that $Mx = x$.*

We say that the node i is *overloaded* if $\sum_{j\in J_i} x_{j-1}v_j > 1$. In this case, the queue length at the node i increases linearly in time. Notice that if all nodes are overloaded, then any steady solution of the reentrant line is a fixed point of the mapping $N\colon x_i \to \tilde{x}_i$, where $x_0 = 1$ and

$$\tilde{x}_s = \frac{x_{s-1}}{\sum_{j\in J_i} x_{j-1}v_j} \quad \text{for} \quad s \in J_i, \quad i = 1, \dots, K. \tag{2.3}$$

The mapping N is well defined for all $x \in \mathbb{R}_+^n$, $x > 0$.

Lemma 2.2. *Let $v_* = \min_{s=1,\dots,n} v_s$, $v^* = \max_{s=1,\dots,n} v_s$ and*

$$Y = \Big\{x \in \mathbb{R}_+^n\colon \sum_{j\in J_i} x_j v_j = 1, \ \ x_s \ge \Big(\frac{v_*}{nv^*}\Big)^s, \ \ i = 1, \dots, K, \ \ s = 1, \dots n\Big\}.$$

Then $NY \subset Y$ and $N^{n+1}x \in Y$ for any $x \in \mathbb{R}_+^n$, $x > 0$.

Proof. Notice that for any $x \in Y$ we have $x \leq (1/v_*, \dots, 1/v_*)^T$. Therefore, if $\tilde{x} = Nx$ then

$$\tilde{x}_s = \frac{x_{s-1}}{\sum_{j\in J_{i(s)}} x_{j-1}v_j} \geq \frac{(v_*/nv^*)^{s-1}}{(1/v_*)\sum_{j\in J_{i(s)}} v_j} \geq \frac{(v_*/nv^*)^{s-1}}{nv^*/v_*} = \left(\frac{v_*}{nv^*}\right)^s \tag{2.4}$$

(since $x_0 = 1$, estimate (2.4) holds for $s = 1$). Hence, $NY \subset Y$, as required. Let

$$Y_0 = \Big\{x \in \mathbb{R}^n_+ \colon \sum_{j\in J_i} x_j v_j = 1,\ \ i = 1, \dots, K\Big\},$$

$$Y_l = \Big\{x \in \mathbb{R}^n_+ \colon \sum_{j\in J_i} x_j v_j = 1,\ \ x_s \geq \left(\frac{v_*}{nv^*}\right)^s,\ \ i = 1, \dots, K,\ \ s = 1, \dots, l\Big\}.$$

Notice that $Nx \in Y_0$ for any $x \in \mathbb{R}^n_+$, $x > 0$, and for any $x \in Y_0$ we have, again, $x \leq (1/v_*, \dots, 1/v_*)^T$. Using estimate (2.4), we get $NY_l \subset Y_{l+1}$. Therefore, $N^{n+1}x \in Y_n = Y$. □

Since Y is compact and convex and N is continuous on Y, we can apply the Brouwer principle once more. Hence, the following lemma holds.

Lemma 2.3. *There exists an $x \in Y \subset \mathbb{R}^n_+$ such that $Nx = x$.*

It is easy to see that any fixed point of the mapping N satisfies a system of K equations.

Lemma 2.4. *For any $x = Nx$ we have*

$$x = (x(1, a), \dots, x(n, a))^T,$$

where $a = (a_1, \dots, a_K)$, $x(0, a) = 1$ and $x(j, a) = x(j-1, a)a_{i(j)}$. The variables $a_1, \dots, a_K$ satisfy a system of K equations:

$$\sum_{j\in J_i} x(j, a)v_j = 1.$$

Proof. Let us denote $a_i = 1/\big(\sum_{j\in J_i} x_{j-1}v_j\big)$, where $x_0 = 1$. Then, (2.3) implies that $x_s = x_{s-1}a_{i(s)}$ for all $s = 1, \dots, n$, and hence $x_j = x(j, a)$. The system of equations arises from the conditions $\sum_{j\in J_i} x_j v_j = 1$, see Lemma 2.3. □

The construction of $x(j, a)$ for a reentrant line from Fig. 1 is illustrated in Fig. 2.

An anologous assertion holds for fixed points of the mapping M, but the corresponding system of equations for variables $a_1, \dots, a_K$ happens to be non-algebraic.

Lemma 2.5. *For any $x = Mx$ we have*

$$x = (x(1, a), \dots, x(n, a))^T,$$

where $a = (a_1, \dots, a_K)$, $x(0, a) = 1$, $x(j, a) = x(j-1, a)a_{i(j)}$ and the variables $a_1, \dots, a_K \leq 1$ satisfy the following conditions:

(1) *if $\sum_{j\in J_i} x(j, a)v_j < 1$ then $a_i = 1$,*
(2) *otherwise $\sum_{j\in J_i} x(j, a)v_j = 1$.*

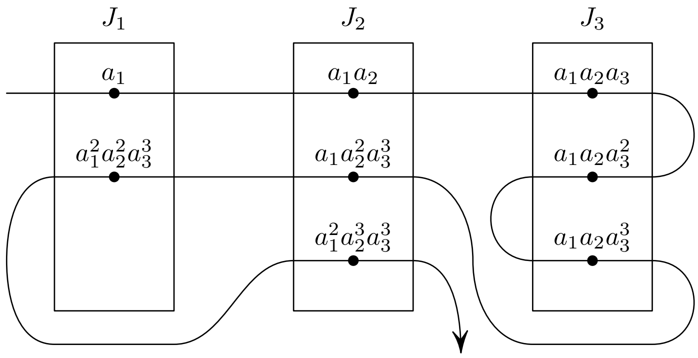

FIGURE 2. A fixed point for reentrant line

In the next section we prove that, for a reentrant line of Kelly type, a steady solution (a fixed point of M) is unique. However, in general, there can exist more than one steady solution of a reentrant line with arbitrary viscosities. In other words, mappings M and N can have different fixed points x and x' for all of which all the nodes are overloaded, that is $a_i < 1$ and $\sum_{j\in J_i} x(j, a)v_j = 1$ for all $1 \le i \le K$.

Example 2.6. *Let us construct two different overloaded solutions of a line with $K = 2$, $n = 5$, $J_1 = \{1, 2, 5\}$ and $J_2 = \{3, 4\}$. Any overloaded solution has the form*

$$x = (a,\, a^2,\, a^2b,\, a^2b^2,\, a^3b^2), \qquad a,\, b \ge 0,$$

and the equalities

$$v_1a + v_2a^2 + v_5a^3b^2 = 1, \qquad v_3a^2b + v_4a^2b^2 = 1 \tag{2.5}$$

must hold. Let us choose $a = b = 1/2$ and $a' = 3/4$, $b' = 1/4$. Now, it is easy to find viscosities v_j, $j = 1, \dots, 5$, such that (2.5) holds together with

$$v_1a' + v_2a'^2 + v_5a'^3b'^2 = 1, \qquad v_3a'^2b' + v_4a'^2b'^2 = 1. \tag{2.6}$$

In particular, one can choose

$$v_1 = \frac{15}{32},\ \ v_2 = \frac{1}{144},\ \ v_5 = \frac{220}{9},\ \ v_3 = \frac{56}{9},\ \ v_4 = \frac{32}{9}.$$

Examples of multiple steady solutions can be constructed on the basis of the following simple observation.

Lemma 2.7. *Let $\alpha, \beta \in \mathbb{R}^n_+$, $\alpha \ne \beta$, $\alpha, \beta > 0$ and $n \ge 2$. Then the relations*

$$(w, \alpha) = 1, \quad (w, \beta) = 1$$

are consistent with respect to $w \in \mathbb{R}^n_+$, $w > 0$ if and only if the vector $\alpha - \beta$ has both positive and negative components, that is, if $(\alpha_i - \beta_i)(\alpha_j - \beta_j) < 0$ for some $1 \le i, j \le n$.

In Example 2.6 we have $\alpha^1 = (a,\, a^2,\, a^3b^2)^T$, $\beta^1 = (a',\, a'^2,\, a'^3b'^2)^T$ and the existence of a required positive vector $w^1 = (v_1,\, v_2,\, v_5)^T$ is ensured by Lemma 2.7, since the difference vector $\alpha^1 - \beta^1 = (a - a',\, a^2 - a'^2,\, a^3b^2 - a'^3b'^2)^T$ has both positive and negative components. The same argument works for $\alpha^2 = (a^2b,\, a^2b^2)^T$, $\beta^2 = (a'^2b',\, a'^2b'^2)^T$, and $w^2 = (v_3,\, v_4)$.

Note that in example (2.5)–(2.6) one of the subsets contains three elements. We show next that, if each subset in the partition of $\{1,\, \dots,\, n\}$ consists of less than 3 elements, then a steady solution is unique. We use the l^1-norm $\|x\| = |x_1| + \dots + |x_n|$ which induces the matrix norm $\|A\| = \max_{j=1,\dots,n}\|Ae_j\|$ on the space of linear operators $A\colon \mathbb{R}^n \to \mathbb{R}^n$, where $\{e_j\}$ are the unit vectors.

Proposition 2.8. *Suppose that $n_i = |J_i| \le 2$ for all $j = 1,\, \dots,\, K$. Then there exists a unique positive fixed point $x^* = Nx^*$ and there exist $\lambda < 1$ and $C > 0$ such that $\|N^m x - x\| \le C\lambda^m$ for all $x \in \mathbb{R}^n_+$, $x > 0$, $m \in \mathbb{N}$.*

Proof. We consider the case of all subsets containing exactly 2 elements. The proof in the case when some of the subsets are singletons is essentially the same. By Lemma 2.2, $N^{n+1}x \in Y$ for all $x \in \mathbb{R}^n_+$, $x > 0$. Therefore, without loss of generality we can assume that $x \ge (\varepsilon,\, \dots,\, \varepsilon)^T$ for some $\varepsilon > 0$. Define $r_s = \ln x_s$. Then $r_0 = 0$ and

$$\tilde r_s = r_{s-1} - \ln\bigl(v_{j_1^i} e^{r_{j_1^i - 1}} + v_{j_2^i} e^{r_{j_2^i - 1}}\bigr)$$

for $s = j_1^i,\, j_2^i$ for all $i = 1,\, \dots,\, K$, where $J_i = \{j_1^i,\, j_2^i\}$. Denote by R the mapping $r \to \tilde r$. Let us find the Jacobi matrix $DR = (\partial \tilde r_s / \partial r_k)$:

$$\frac{\partial \tilde r_s}{\partial r_k} = \begin{cases} 0, & s \in J_i, \quad k \ne j_1^i - 1,\, j_2^i - 1, \\ \begin{cases} 1 - \alpha_{j_1^i}, & s = j_1^i, \\ -\alpha_{j_1^i}, & s = j_2^i, \end{cases} & k = j_1^i - 1, \\ \begin{cases} -\alpha_{j_2^i}, & s = j_1^i, \\ 1 - \alpha_{j_2^i}, & s = j_2^i, \end{cases} & k = j_2^i - 1, \end{cases}$$

where

$$\alpha_s = \frac{v_s e^{r_{s-1}}}{v_{j_1^i} e^{r_{j_1^i - 1}} + v_{j_2^i} e^{r_{j_2^i - 1}}}, \qquad s \in J_i, \quad i = 1,\, \dots,\, K.$$

Clearly, $\alpha_s \le 1$. One easily sees that, for any e_i and r, the inequality $\|DR(r)e_i\| \le 1$ holds. Therefore, for any r^1, r^2 we have

$$\|Rr^2 - Rr^1\| = \left\| \int_0^1 DR(\gamma(s))\,\gamma'(s)\,ds \right\| \le \|r^2 - r^1\|, \tag{2.7}$$

where $\gamma(s) = (1 - s)r^1 + sr^2$.

Let $x^* = Nx^*$ be a fixed point of N. Then the vector r^* with components $r_i^* = \ln x_i^*$ is a fixed point for R. Let $P = |DR(r^*)|^T$, where $|\cdot|$ is applied to DR component-wise and ^{T} denotes transposition. Then P is a transition matrix for a finite Markov chain with states $\{1,\, \dots,\, n\}$ which terminates on hitting the state n, i.e., the row corresponding to the state n consists of zeros. Since the state n is absorbing, the spectral radius of P is strictly less than 1. Hence, the spectral radius of $DR(r^*)$ is also strictly less than one. Therefore, r^* is locally stable and attracts with exponential rate all points from some neighbourhood of r^*. This together with (2.7) implies that r^* attracts all $r \in \ln Y$ with exponential rate uniformly in $r \in \ln Y$. Since $R^{n+1} r \in \ln Y$ for any $r \in \mathbb{R}^n$, the uniform exponential convergence takes place for all $r \in \mathbb{R}^n$ as required.

If $n_i = 2$ not for all i, then one can still show that (2.7) holds. Furthermore, in this case the corresponding Markov chain with transition matrix $P = |DR(r^*)|^T$ has more than one absorbing state. Hence, all the above arguments hold and there is the global convergence to a unique stationary point r^*. $\square$

An analogous assertion holds for the mapping M.

Proposition 2.9. *Suppose that $n_i = |J_i| \leq 2$ for all $j = 1, \ldots, K$. Then there exists a unique fixed point $x^* = Mx^*$ and there exist $\lambda < 1$ and $C > 0$ such that $\|M^m x - x\| \leq C\lambda^m$ for all $x \in \mathbb{R}^n_+$, $m \in \mathbb{N}$.*

Proof. We give only a sketch of the proof. Again, consider the case of all subsets containing exactly 2 elements. As in the proof of Proposition 2.8, we pass to the logarithmic coordinates $r = \ln x$. Notice that one can represent $R : r \to \tilde{r}$ as the composition $R = LS$, where S is a shift operator

$$S(r_1, \ldots, r_n) = (0, r_1, \ldots, r_{n-1})$$

and L is the normalization operator which acts on each node J_i independently of other nodes:

$$L(a, b) = \begin{cases} (a, b), & \text{if } W = v_a e^a + v_b e^b \leq 1, \\ (a - W, b - W), & \text{if } W = v_a e^a + v_b e^b > 1, \end{cases}$$

where $(a, b) = (r_{j_1^i - 1}, r_{j_2^i - 1})$. Let us denote

$$\Omega_{\leq} = \{(a, b) \colon v_a e^a + v_b e^b \leq 1\} \quad \text{and} \quad \Omega_{>} = \{(a, b) \colon v_a e^a + v_b e^b > 1\}.$$

It is easy to see that $\Omega_{\leq}$ is a convex domain. Denote by $m_1 = (a_1, b_1)$ and $m_2 = (a_2, b_2)$ two points in $\mathbb{R}^2$. Clearly, if $m_1, m_2 \in \Omega_{\leq}$, then $Lm_1 = m_1$, $Lm_2 = m_2$ and $\|Lm_2 - Lm_1\| = \|m_2 - m_1\|$. If $m_1 \in \Omega_{\leq}$, $m_2 \in \Omega_{>}$, then the interval $\gamma(s) = (1 - s)m_1 + sm_2$, $0 \leq s \leq 1$ meets $\partial\Omega_{\leq}$ at a single point m_3. Notice that an open interval with endpoints m_2, m_3 belongs to $\Omega_{>}$. Similarly to (2.7) we have $\|Lm_2 - Lm_3\| \leq \|m_2 - m_3\|$. Therefore,

$$\begin{aligned} \|Lm_2 - Lm_1\| &\leq \|Lm_2 - Lm_3\| + \|Lm_3 - Lm_1\| \\ &\leq \|m_2 - m_3\| + \|m_3 - m_1\| = \|m_2 - m_1\|. \end{aligned}$$

The case $m_1, m_2 \in \Omega_{>}$ can be considered analogously. The only difference is connected with the fact that the interval $\gamma(s)$ may intersect $\partial\Omega_{\leq}$ in two points. Therefore,

$$\|Rr^2 - Rr^1\| \leq \|r^2 - r^1\|.$$

The remaining part of the proof is almost the same as for Proposition 2.8. One needs only to deal carefully with the case when M is not differentiable at the fixed point r^*. $\square$

The following corollary is an immediate consequence of Proposition 2.9.

Corollary 2.10. *Suppose that $n_i = |J_i| \leq 2$ for all $j = 1, \ldots, k$. Then there exists a unique steady solution satisfying* (1.1).

We finish this section by the following simple proposition. It shows that the nonuniqueness example (2.5)–(2.6) is in some sense minimal.

Proposition 2.11. *If $n \leq 4$ then there exists a unique steady solution of the reentrant line.*

The assertion essentially follows from the previous corollary. One just has to consider separately three cases where one of the subsets contains 3 elements. It is easy to check that in all these cases the assertion holds.

3. Reentrant lines of Kelly type

In this section we study reentrant lines of Kelly type ([5], see also Introduction), that is, we assume

$$v_j = V_i > 0 \quad \text{for all } j \in J_i, \quad i = 1, \ldots, K. \tag{3.8}$$

Theorem 3.1 (Main Theorem)**.** *Let* (3.8) *hold. Then for any steady input there exists a unique steady solution of the reentrant line.*

Proof. By Lemma 2.1, it suffices to prove the uniqueness of the vector x satisfying conditions (1.1) from Introduction. Let us consider the cube $\overline{Q}$

$$\overline{Q} = \{a \in \mathbb{R}^K : \; 0 \leq a_i \leq 1, \; i = 1, \ldots, K\}$$

and a vector $a \in \overline{Q}$. The component a_i, $i = 1, \ldots, K$, will be interpreted as the ratio of the outcoming flow from node i to its incoming flow. Note that, since only steady flows are considered, this ratio is the same for any particular class $j \in J_i$, that is,

$$x_j = a_{i(j)} x_{j-1}, \quad j = 1, \ldots, n,$$

where x_j is the rate of the flow leaving class j, cf. Lemma 2.5.

First, let us find the actual rate $x(j, a)$ for a given $a \in \overline{Q}$. We set $x(0, a) = 1$ for each $a \in \overline{Q}$ and use the recurrent relation

$$x(j, a) = x(j-1, a) a_{i(j)}$$

to define $x(j, a)$, $j = 1, \ldots, n$. Hence, $x(j, a)$ is a monomial of the form

$$x(j, a) = \prod_{i=1,\ldots,K} a_i^{p_i(j)}, \quad j = 0, 1, \ldots, n, \quad a \in \overline{Q},$$

where integer-valued functions $p_i(j)$ of integer argument $j \in \{0, \ldots, n\}$ are defined by the relations

$$p_i(0) = 0, \qquad p_i(j) = \begin{cases} p_i(j-1) + 1 & \text{if } j \in J_i, \\ p_i(j-1) & \text{otherwise,} \end{cases}$$

for $i = 1, \ldots, K$, $j = 1, \ldots, n$. Let us extend the domain of $p_i(\cdot)$ to the real interval $[0, n]$ setting

$$p_i(j + \alpha) = \alpha p_i(j+1) + (1-\alpha) p_i(j), \quad \alpha \in [0, 1], \; j = 0, \ldots, n-1, \; i = 1, \ldots, K.$$

The rate of the total outcoming flow from node i is found as

$$F_i(a) = \sum_{j \in J_i} x(j,\, a), \quad i = 1,\, \dots,\, K,\;\; a \in \overline{Q}.$$

The continuous map $F\colon \overline{Q} \to \mathbb{R}_+^K$, $F(a) = (F_1(a),\, \dots,\, F_K(a))$ is nondecreasing with respect to the cone $\mathbb{R}_+^K$, i.e., $a \geq b$ implies $F(a) \geq F(b)$, where the inequalities hold component-wise. Let us denote

$$b_i = \frac{1}{V_i}.$$

It is easy to see that any steady solution with $x_0 = 1$ is given by a vector

$$(x_0,\, x_1,\, \dots,\, x_n) = (1,\, x(1,\, a),\, \dots,\, x(n,\, a)),$$

where a satisfies the conditions

$$F_i(a) \leq b_i, \quad i = 1,\, \dots,\, K, \tag{3.9}$$

and

$$(b_i - F_i(a))(1 - a_i) = 0, \quad i = 1,\, \dots,\, K. \tag{3.10}$$

We will prove the uniqueness of such a vector $a \in \overline{Q}$ which is equivalent to the assertion of the theorem. Notice that all the components of any steady solution are strictly positive. It follows that there exists a constant $\delta > 0$ which depends only on $V_1,\, \dots,\, V_K$ and n such that any vector a corresponding to a steady solution belongs to the cube

$$Q = \{a \in \mathbb{R}^K\colon\;\; \delta \leq a_i \leq 1,\;\; i = 1,\, \dots,\, K\}.$$

Let us fix an $a \in Q$ and find the Jacobi matrix

$$DF(a) = \left(\frac{\partial F_i(a)}{\partial a_k}\right)_{i,k=1,\dots,K}.$$

Since, for any $j = 1,\, \dots,\, n$,

$$\frac{\partial}{\partial a_k} \prod_{m=1,\dots,K} a_m^{p_m(j)} = \frac{p_k(a)}{a_k} \prod_{m=1,\dots,K} a_m^{p_m(j)},$$

we have

$$\frac{\partial F_i(a)}{\partial a_k} = \sum_{j \in J_i} \frac{\partial x(j,\, a)}{\partial a_k} = \sum_{j \in J_i} \frac{x(j,\, a)}{a_k}\, p_k(j). \tag{3.11}$$

Let us consider two cases. First, let $j \in J_i$. Since $\dot{p}_i(t) \equiv 1$ and $\dot{p}_k(t) \equiv 0$ for $j - 1 < t \leq j$ and $k \neq i$, we get the following equality:

$$x(j,\, a)\, p_k(j) = \begin{cases} \displaystyle\int_{j-1}^{j} x(j,\, a)\, p_k(t)\, \dot{p}_i(t)\, dt & \text{if } i \neq k, \\ \displaystyle\int_{j-1}^{j} x(j,\, a)\, p_k(t)\, \dot{p}_i(t)\, dt + \frac{x(j,\, a)}{2} & \text{if } i = k. \end{cases}$$

If $j \notin J_i$, then $\dot{p}_i(t) \equiv 0$ for $j - 1 < t \leq j$ and, hence,

$$\int_{j-1}^{j} x(j,\, a)\, p_k(t)\, \dot{p}_i(t)\, dt = 0.$$

Therefore, (3.11) can be written as

$$\frac{\partial F_i(a)}{\partial a_k} = \frac{1}{a_k}\int_0^n x(t,\,a)\,p_k(t)\,\dot p_i(t)\,dt + \Lambda_{ik}(a), \tag{3.12}$$

where $x(t,\,a) = x(\lceil t\rceil,\,a)$ (here $\lceil t\rceil$ is the least integer $\bar t$ such that $t \le \bar t$) and $\Lambda(a) = \{\Lambda_{ik}\colon 1 \le i,\,k \le K\}$ is a positive definite diagonal matrix:

$$\Lambda(a) = \operatorname{diag}(\lambda(a)), \qquad \lambda(a) = (\lambda_1(a),\,\ldots,\,\lambda_K(a)),$$

and $\lambda_i(a) = \frac{1}{2a_i}\sum_{j\in J_i} x(j,\,a) = \frac{F_i(a)}{2a_i} > 0$ for any $a > 0$, $i = 1,\,\ldots,\,K$.

Let us make the change of variables $r_i = \ln a_i$, $i = 1,\,\ldots,\,K$. This change transfers the cube Q onto the cube

$$Q' = \{r \in \mathbb{R}^K\colon\ \ln\delta \le r_i \le 0,\ \ i = 1,\,\ldots,\,K\}.$$

Let us use the notation $\widetilde F(r) = F(a)$, where $a_i = e^{r_i}$, $i = 1,\,\ldots,\,n$, and note that

$$\frac{\partial \widetilde F_i(r)}{\partial r_k} = a_k\,\frac{\partial F_i(a)}{\partial a_k}$$

where $1 \le i,\,k \le K$ and $r = \ln a$ (component-wise). We get

$$\frac{\partial \widetilde F_i(r)}{\partial r_k} = \int_0^n x(t,\,a)\,p_k(t)\,\dot p_i(t)\,dt + \widetilde\Lambda_{ik}(a),$$

where

$$\widetilde\Lambda(r) = \operatorname{diag}(\lambda'(r)), \qquad \lambda'(r) = \left(\frac{\widetilde F_1(r)}{2},\,\ldots,\,\frac{\widetilde F_K(r)}{2}\right).$$

It is important for what follows that $x(j,\,a)$ is a non-increasing positive function of j for each $a \in Q$. Let us prove that for any vector $r \in Q'$ the Jacobi matrix $D\widetilde F(r)$ is positive definite. Notice that

$$x(t,\,a) = \sum_{j=1}^n (x(j,\,a) - x(j+1,\,a))1_{\{t\in(0,j]\}}, \tag{3.13}$$

where $x(n+1,\,a) \equiv 0$ and

$$1_{\{t\in(0,j]\}} = \begin{cases} 1, & t \in (0,\,j],\\ 0, & t \notin (0,\,j].\end{cases}$$

Using (3.12) and (3.13) we get

$$\begin{aligned}
\langle D\widetilde F(r)x,\,x\rangle &= \sum_{i=1}^K\sum_{k=1}^K\Bigl(\int_0^n x(a,\,t)\,p_k(t)\,\dot p_i(t)\,dt\Bigr)x_k + \sum_{i=1}^K x_i^2\widetilde F_i(r)\\
&= \int_0^n x(a,\,t)\,\langle p(t),\,x\rangle\,\frac{d}{dt}\,\langle p(t),\,x\rangle\,dt + \sum_{i=1}^K x_i^2\widetilde F_i(r)\\
&= \sum_{j=1}^n (x(j,\,a) - x(j+1,\,a))\int_0^j \langle p(t),\,x\rangle\,\frac{d}{dt}\,\langle p(t),\,x\rangle\,dt + \sum_{i=1}^K x_i^2\widetilde F_i(r)\\
&= \sum_{j=1}^n (x(j,\,a) - x(j+1,\,a))\,\frac{\langle p(j),\,x\rangle^2}{2} + \sum_{i=1}^K x_i^2\widetilde F_i(r) > 0
\end{aligned} \tag{3.14}$$

for any $x \in \mathbb{R}^K$, $x \neq 0$.

Now, it is easy to prove that

$$\langle \widetilde{F}(r') - \widetilde{F}(r),\, r' - r \rangle > 0 \tag{3.15}$$

whenever r, $r' \in Q'$ and $r \neq r'$. Indeed,

$$\widetilde{F}(r') - \widetilde{F}(r) = \int_0^1 D\widetilde{F}(\gamma(s))\,\gamma'(s)\,ds,$$

where $\gamma(s) = (1-s)r + sr'$ and $\gamma'(s) = r' - r$. Since (3.14) holds for all $r \in Q'$ and Q' is a convex set, we get

$$\langle \widetilde{F}(r') - \widetilde{F}(r),\, r' - r \rangle = \int_0^1 \langle D\widetilde{F}(\gamma(s))(r' - r),\, r' - r \rangle\, ds > 0,$$

as required.

Finally, let us suppose that a and a', $a \neq a'$, from Q generate two steady solutions of the reentrant line. Let $r = \ln a$ and $r' = \ln a'$. Let us demonstrate that

$$\langle \widetilde{F}(r') - \widetilde{F}(r),\, r' - r \rangle \leq 0$$

which is a contradiction to (3.15).

It suffices to show that $(\widetilde{F}_i(r') - \widetilde{F}_i(r))(r'_i - r_i) \leq 0$ for any $i = 1, \ldots, n$. Let us consider four cases. If $\widetilde{F}_i(r') = b_i$ and $\widetilde{F}_i(r) = b_i$ then $(\widetilde{F}_i(r') - \widetilde{F}_i(r))(r'_i - r_i) = 0$. If $\widetilde{F}_i(r') < b_i$ and $\widetilde{F}_i(r) < b_i$ then $a_i = a'_i = 1$, hence $r_i = r'_i = 0$ and, again, $(\widetilde{F}_i(r') - \widetilde{F}_i(r))(r'_i - r_i) = 0$. If $\widetilde{F}_i(r') = b_i$ and $\widetilde{F}_i(r) < b_i$ then $(\widetilde{F}_i(r') - \widetilde{F}_i(r)) > 0$ and $r_i = 0$. Since $r'_i \leq 0$, we get $r'_i - r_i \leq 0$ and $(\widetilde{F}_i(r') - \widetilde{F}_i(r))(r'_i - r_i) \leq 0$. The case $\widetilde{F}_i(r') < b_i$ and $\widetilde{F}_i(r) = b_i$ is equivalent to the previous one. □

Remark. Theorem 3.1 can be extended to a more general case of a reentrant line with additional inflows at each node and a fixed fraction of the flow leaving the system at each node. It also can be extended to the case of a finite number of reentrant lines. In this situation one has to concatenate all the reentrant lines into a single reentrant line and assume that the total flow leaves the system at each conjunction point and a new flow comes from the outside.

We thank M. Blank, V. Kleptsyn and S. Fomin for helpful discussions.

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