Large transportation networks with finite-dimensional state space. Asimptotic approach.

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Consider an open network consisting of N nodes (stations) and $V(N)$ cars, which circulate among the stations. The network is divided into n clusters. As N increases the number of clusters does not change. Cluster j contains $d_j^N N$ stations, $\sum_{j=1}^n d_j^N = 1$. There exists $d_j > 0$ so that $\sqrt{N}(d_j^N - d_j) \xrightarrow{N\to\infty} 0$ for $j = \overline{1,n}$. Later j and v denote cluster numbers and they take values from 1 to n. The arrival process to a station of cluster j is a Poisson one with the rate λ_j.

If arrived customer finds a car at the node he takes it to reach his destination. Cluster v is chosen via routing matrix $P = \{p_{jv}\}_{j,v=\overline{1,n}}$. The destination station in the cluster v is chosen uniformly. After having reached their destinations, customers leave the network.

The fully symmetric network was considered in [1]. Our model represents an asymmetric generalization of [1]. The travel time from a station in cluster j to a station in cluster v is exponentionally distributed with a parameter μ_{jv}. The travel time between two stations in cluster j is also exponentionally distributed with a parameter μ_{jj}. A customer which upon arrival does not find an available car and finds free waiting place joins the queue. Otherwise he leaves the network. Capacities of waiting rooms for customers in cluster v are supposed limited by k_v for each station. There are m_v parking lots for servers at each station. If a car finds a station in cluster v empty and number of cars at the station is less then m_v, it stops and waits for the next customer at this node. Otherwise, it is directed to cluster l via routing matrix $\widetilde{P} = \{\tilde{p}_{vl}\}_{v,l=\overline{1,n}}$. The car chooses a station inside cluster l uniformly.

Initially, each station in cluster j has r_j servers where r_j is integer from 0 to m_j, $V(N) = (r_1 d_1^N + \cdots + r_n d_n^N)N$. Matrix $P\widetilde{P}$ is assumed to be ergodic.

Let $x_{j,i}(t)$ be a fraction of nodes in state i at cluster j among all N stations, $\sum_{i=-k_j}^{m_j} x_{j,i}(t) = d_j^N$. Let $\widetilde{M}_{jv}(t)$ denotes the number of cars driving from cluster j to cluster v at the moment t. One can describe the state of the network as a vector $x \in \mathbf{R}^\alpha$ with $\alpha = n^2 + \sum_{v=1}^n (k_v + m_v + 1)$ components: $x = (M_{11}, M_{12}, \ldots, M_{1n}, M_{21}, \ldots, M_{nn}, x_{1,-k_1}, x_{1,-k_1+1}, \ldots, x_{1,m_1}, x_{2,-k_2}, \ldots, x_{n,m_n})$, where $M_{jv}(t) = \widetilde{M}_{jv}(t)/N$. It is clear that the stochastic process $X_t^N = x(t)$ is the ergodic Markov chain.

Let $x_t(x)$ be a solution of the following system of ordinary differential equations with the initial point $x_0(x) = x$:

$$\begin{aligned}
\dot{x}_{j,-k_j} &= (\lambda_j d_j x_{j,-k_j+1} - M^j x_{j,-k_j})/d_j, \\
\dot{x_{j,i}} &= (\lambda_j d_j x_{j,i+1} - (\lambda_j d_j + M^j) x_{j,i} + M^j x_{j,i-1})/d_j, \\
\dot{x}_{j,m_j} &= (-\lambda_j d_j x_{j,m_j} + M^j x_{j,m_j-1})/d_j, \\
\dot{M}_{jv} &= p_{jv}\left(\lambda_j d_j S_j^+ + M^j S_j^-\right) - \mu_{jv} M_{jv} + d_j M^j x_{j,m_j} \tilde{p}_{jv}
\end{aligned}$$

for $j, v = \overline{1,n}$, $i = \overline{-k_j+1, m_j-1}$. Here $S_j^+ \equiv d_j \sum_{i=1}^{m_j} x_{j,i}$, $S_j^- \equiv d_j \sum_{i=-k_j}^{-1} x_{j,i}$, $M^j \equiv \sum_{l=1}^n \mu_{lj} M_{lj}$. Let ε_x denote a distribution which is concentrated in a point $x \in \mathbf{R}^\alpha$.

Theorem. *If* $X_0^N \to \varepsilon_x$ *weakly then*

(i) $\sup\limits_{0 \le s \le t} |X_s^N - x_s(x)| \xrightarrow{P} 0$ *for all* $t \ge 0$ *as* $N \to \infty$,

(ii) proceses $\sqrt{N}(X_t^N - x_t(x))$ *weakly converges to a continuous process* Y *with independent increments and covariation function* $\hat{C}(x)$. *Here* $\hat{C}(x) = \int\limits_0^t c(x_s(x))ds$ *and* c *is a matrix function which can be written in explicit form.*

The results obtained allow to study some parameters of X_t^N with the help of non-linear dynamical system $x_t(x)$.

[1] L.G. Afanassieva, G. Fayolle, S.Yu. Popov. *Models for transportation networks.*