Non – homogeneous zero – range interaction processes

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Abstract

In this résumé we formulate the results concerning the limiting behaviour (as time $t \to \infty$) of a non–homogeneous zero–range interaction particle system. Similar (homogeneous) processes were introduced by Spitzer (1970) and investigated by E. Waymire (1980), E. D. Andjel (1982), A. Galves and H. Guiol (1997). As this system is not ergodic we describe the class of its stationary distributions and explore the character of non–stability in particular cases.

Consider the following particle system: a number of indistinguishable particles is located in a countable set of sites J. Transitions of particles at a site i are made after random periods of time τ — i. i. exponentially d. r. v. with parameter $\gamma_i \; \forall\, i \in J$: if site i is not empty then one (and only one) particle chosen randomly at this site instantly moves to any site j w. p. p_{ij}. Transitions are made independently for any $i \in J$. Probabilities of jumps from i to j form together matrix $P = (p_{ij})_{i,j \in J}$, $\forall\, i \in J \; \sum\limits_{j \in J} p_{ij} = 1$. One can describe the state of the system by process $\eta(t) = (\eta_i(t); i \in J)$, $t \geq 0$, where $\eta_i(t)$ is a number of particles in cell i at time t. Denote $\gamma = \sum\limits_{i \in J} \gamma_i \leq \infty$. Let $\overline{\mathbf{Z}}_+ = \{0, 1, 2, \ldots\} \cup \{\infty\}$. Introduce $\mathcal{B}$ — a σ–algebra generated by open sets in the product topology on the state space $\mathbf{W} = \overline{\mathbf{Z}}_+^J$ of our system.

Theorem 1 (Existence) *If* $\sup\limits_{i \in J} \gamma_i < \infty$, $\sup\limits_{i \in J} \sum\limits_{j \in J} \gamma_j p_{ji} < \infty$ *then there exists unique Feller process* $\eta(t) : (\Omega, \Im, \mathbf{P}) \to (\mathbf{W}, \mathcal{B})$ *that describes our system,* $(\Omega, \Im, \mathbf{P})$ *— some probability space.*

Suppose the Markov chain with state space J and transition matrix P to be homogeneous, aperiodic, and irreducible. Let us state
Conjecture A: There exists a non–trivial invariant measure $\pi = (\pi_i)_{i \in J}$ for P (not necessarily finite).
Conjecture B: A countable Markov chain with transition matrix P is transient.

For theorems 2 and 3 suppose $\sup\limits_{i \in J} \pi_i/\gamma_i < \infty$ and denote $\rho_{\max} = 1/(\sup\limits_{i \in J} \pi_i/\gamma_i)$. For any $\rho \in [0; \rho_{\max}]$ and $\rho = \infty$ introduce product measures $L_{\rho a}(\cdot)$ on $(\mathbf{W}, \mathcal{B})$ with marginal factors $l^i_{\rho a}(\cdot)$, $i \in J$ defined in the following two cases: 1) $\rho \in [0; \rho_{\max}]$, $\forall i$ or $\rho = \rho_{\max}$, $\rho(\pi_i/\gamma_i) < 1$; 2) $\rho = \rho_{\max}$, $\rho(\pi_i/\gamma_i) = 1$ or $\rho = \infty$, $\forall i$:

$$1) \; l^i_{\rho a}(k) = \begin{cases} (1 - \rho(\pi_i/\gamma_i)) \left(\rho(\pi_i/\gamma_i)\right)^k, & k \in \mathbf{Z}_+, \\ 0, & k = \infty, \end{cases} \qquad 2) \; l^i_{\rho a}(k) = \begin{cases} 0, & k \in \mathbf{Z}_+, \\ 1, & k = \infty. \end{cases}$$

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Theorem 2 (Invariant measures) *Suppose that conjecture A holds. Then the closed convex hull of the set of measures* $\{L_{\rho a}(\cdot): \ \rho \in [0;\rho_{\max}], \ \rho = \infty\}$ *belongs to the class* $\mathcal{M}$ *of all invariant measures for Markov process* $\{\eta(t)\}_{t\geq 0}$.

Theorem 3 (Clustering) *Suppose that conjecture A holds and* $\sum\limits_{i\in J} \pi_i/\gamma_i < \infty$. *Let* $j_0 \in J$ *satisfies* $\pi_{j_0}/\gamma_{j_0} = \max\limits_{i\in J} \pi_i/\gamma_i$. *Then* $\forall\ k \in \mathbf{Z}_+ \ \ \forall \eta^0 : \sum\limits_{i\in J} \eta_i^0 = \infty$

$$\lim_{t\to\infty} \mathbf{P}\left\{\eta_{j_0}(t) > k \mid \eta(0) = \eta^0\right\} = 1.$$

Theorem 4 (Devastation) *Suppose that one of the following conditions holds:*

1. $\gamma < \infty$ *and conjecture B.*
2. $\gamma < \infty$ *and* $\sum\limits_{i\in J} p_{ij} \leq 1$.
3. $\sum\limits_{i\in J} p_{ij} \leq 1$ *and there exists a sequence of finite sets* $\{J_n\}$, $J_1 \subset J_2 \subset \ldots$, $\bigcup\limits_n J_n = J$ *such that* $\sup\limits_{j\in J\setminus J_n} \gamma_j \longrightarrow 0$ *as* $n \to \infty$ *and matrices* $P^{(n)} = (p_{ij})_{i,j\in J_n}$ *are irreducible.*
4. *Let* $\sum\limits_{i\in J} p_{ij} \leq 1$. *Introduce the terminating Markov chain* Y_m *with state space* J *and transition matrix* $P^T = (p_{ji})_{i,j\in J}$. *Denote by* $\bar{P}_j(J)$ *the probability of terminating for* Y_m *provided that* $Y_0 = j$. *Let* $\bar{P}_j(J) = 1$ *for all* $j \in J$ *and* $\sup\limits_{j\in J} \gamma_j < \infty$.

Then for any $\eta(0) \in \mathbf{Z}_+^J$ *the process* $\eta(t) \to 0$ *weakly as* $t \to \infty$.

Theorem 5 (Stochastical boundedness) *If* $\exists i_0 \in J$: $\gamma_{i_0} > \sum\limits_{j\in J} \gamma_j p_{ji_0}$ *and* $\eta_{i_0}(0) \in \mathbf{Z}_+$ *then* $\eta_{i_0}(t)$ *is stochastically bounded.*

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