Mean-field approximation for stochastic transportation network and stability of dynamical system

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Abstract

A queuing system is considered, with a virtual node, N nodes, and rN servers. At each node (of these N nodes) the arrivals of particles form a Poisson flow of rate $\lambda(t)$. For an empty node a particle leaves the system. A server at the node takes the particle and moves to a random node. Travelling time is exponential of mean 1. The number of servers at each node (of these N nodes) is bounded by m.

We discuss the property of stability for the limiting deterministic process as $N \to \infty$. Furthermore, we consider a particular queuing system with dynamical routing.

1 Introduction

Consider a network, consisting of N nodes, a virtual node and rN servers. At each node (of these N nodes) the arrivals of particles form a Poisson flow of rate $\lambda(t)$. If a particle arrives at an empty node then the particle leaves the system. Otherwise, if there is a server at the node then the server takes the particle and jumps to the virtual node. At this node the server waits for an exponential time of mean 1. After the server jumps to a random node with uniform distribution. If the number of servers at the chosen node equals m then the server waits for the following attempt at the virtual node. The non-negative number of servers at each node (except virtual) is bounded by m. Consider the fractions $f_k = n_k/N$, $V = W/N$, where n_k is the (random) number of nodes with k servers and W is the number of servers at the virtual node. It is more convenient to regard the tail probabilities $u_k = \sum_{i=k}^{m} f_i$.

The state space of the corresponding Markov process $U_N(t) = (u_k(t),\, V(t))$ is the set X_N of all vectors $u = (u_1,\, \ldots,\, u_m,\, V)^T$ in $(1/N)\mathbb{Z}_+^{m+1}$ such that

$$1 = u_0 \geq u_1 \geq \ldots \geq u_m \geq 0,\ V \geq 0,\ u_1 + \ldots + u_m + V = r. \tag{1.1}$$

The generator of $U_N(t)$ is the operator $A_N(t)$ acting on functions and given by

$$A_N(t)f(u) = N\lambda(t)\sum_{k=1}^{m-1}(u_k - u_{k+1})[f(u - \frac{e_k}{N} + \frac{e_{m+1}}{N}) - f(u)]+$$

$$+N\lambda(t)u_m[f(u - \frac{e_m}{N} + \frac{e_{m+1}}{N}) - f(u)]+ \tag{1.2}$$

$$+NV\sum_{k=1}^{m}(u_{k-1}-u_k)[f(u+\frac{e_k}{N}-\frac{e_{m+1}}{N})-f(u)],$$

where e_k denotes a vector with the component of number k equal to 1 and others equal to 0.

The mean-field approximation suggests that the whole process $U_N(t)$ is asymptotically deterministic as $N\to\infty$. More precisely, let X denote the set of all $\mathbb{R}^{m+1}$ vectors defined by (1.1). Then, if the distribution of the initial state $U_N(0)$ converges to the Dirac delta-measure concentrated at some point $g\in X$, the distribution of $U_N(t)$ is concentrated on the orbit $u(t)\in X$ as $N\to\infty$ where $u(t)$ is the solution of the following system of differential equations (mean-field equations)

$$\dot{u}=f(u(t),\lambda(t)), \tag{1.3}$$

where

$$\begin{array}{l} f_i(u,\lambda)=\lambda(u_{i+1}-u_i)+V(u_{i-1}-u_i),\ i=1,\dots,m-1, u_0=1,\\ f_m(u,\lambda)=-\lambda u_m+V(u_{m-1}-u_m),\\ f_{m+1}(u,\lambda)=\lambda u_1-V(1-u_m). \end{array} \tag{1.3$'$}$$

System (1.3) is non-linear and its right-hand side depends on time. We study system (1.3) with the help of theorem 3.3. Perhaps theorem 3.3 is the most marvelous result of the present paper. This theorem is of great practical and theoretical interest.

2 Results

By definition, put $\|u\|=|u_1|+\ldots+|u_m|+|V|$. By $\lambda(t)$ denote a function of variable $t\in\mathbb{R}$ such that

$$\inf_{t\geq\tau}\lambda(t)=\bar{\lambda}>0,\ \sup_{t\geq\tau}\lambda(t)<\infty. \tag{2.1}$$

Theorem 2.1. *Suppose the function $\lambda(t)$ is piecewise continuous; then*

(a) For any $g\in X$ and $\tau\in\mathbb{R}$ there exists a unique solution $u(t,\tau,g)$ of the Cauchy problem (1.3) such that $u(\tau,\tau,g)=g$. For any $t\geq\tau$ the solution $u(t,\tau,g)\in X$.

(b) There exists a positive number γ such that $\forall g,\ g'\in X$, $\forall t\geq\tau$

$$\|u(t,\tau,g)-u(t,\tau,g')\|\leq const\cdot\exp\left(-\gamma(t-\tau)\right).$$

(c) Now suppose that $\lambda(t)$ is T-periodic. Then, there exists a unique point $g^*\in X$ such that $u(t,\tau,g^*)$ is a T-periodic solution and $\forall g\in X$

$$\|u(t,\tau,g)-u(t,\tau,g^*)\|\leq const\cdot\exp\left(-\gamma(t-\tau)\right);$$

(d) Further assume that the function $\lambda(t)$ is constant. Now there exists a unique point $g^*\in X$ such that $u(t,\tau,g^*)=g^*$ and $\forall g\in X$

$$||u(t,\tau,g)-g^*||\leq const\cdot\exp\left(-\gamma(t-\tau)\right).$$

The point g^* of the assertion (d) is easily computed (see [1]). Let $T_N(t,\tau)$ be a family of operators such that

$$T_N(t,\tau)f(g)=\mathbf{E}\big(f(U_N(t))\mid U_N(\tau)=g\big),\ g\in X_N. \tag{2.1}$$

Theorem 2.2. *Suppose the function $\lambda(t)$ is piecewise constant; then $\forall f\in C(X)$*

$$\lim_{N\to\infty}\sup_{g\in X_N}|T_N(t,\tau)f(g)-f(u(t,\tau,g))|=0. \tag{2.2}$$

The last limit is uniform in $t\in I$ where the interval $I\subset\{t\geq\tau\}$ is bounded.

Let ε_g be a measure on X such that $\varepsilon_g(\{g\}) = 1$ and $\varepsilon_g(X \setminus \{g\}) = 0$.

Theorem 2.3. *Suppose the function $\lambda(t)$ is piecewise constant and $U_N(\tau) \to \varepsilon_g$ weakly; then*

$$\forall t \sup_{0 \le s \le t} \|U_N(t) - u(t, \tau, g)\| \to 0 \text{ in probability.}$$

Theorem 2.4. *Suppose the function $\lambda(t)$ is constant. Let μ_N be a unique stationary distribution of Markov process U_N; then $\mu_N \to \varepsilon_{g^*}$ weakly.*

3 Auxiliary statements

3.1 Coefficient of ergodicity

Consider a linear space $\mathbb{R}^n$ supplied with the norm $\|x\| = |x_1| + \ldots + |x_n|$ where $x \in \mathbb{R}^n$. Let $\{e_k\}$ be the set of n unit vectors. The space of linear maps $A : \mathbb{R}^n \to \mathbb{R}^n$ is supplied with the following norm

$$\|A\| = \sup_{x \in \mathbb{R}^n, \|x\|=1} \|Ax\| = \sup_i \|Ae_i\|.$$

By definition, put $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \ge 0\}$. A linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ is *Markov* if $A\mathbb{R}^n_+ \subseteq \mathbb{R}^n_+$ and $(1, \ldots, 1)A = (1, \ldots, 1)$. Consider the linear subspace $L = \{x \in \mathbb{R}^n \mid x_1 + \ldots + x_n = 0\}$. If the linear map A is Markov, then the following norm is well defined.

$$\|A\|_L = \sup_{x \in L, \|x\|=1} \|Ax\| = \max_{i,j} \|A(e_i - e_j)\|/2. \tag{3.1}$$

The number $k(A) = \|A\| - \|A\|_L$ is called the *coefficient of ergodicity*. If $B = bA$, $b \ge 0$, and the map A is Markov, then $k(B) = b\,k(A)$.

A linear map A is *non-negative* (*positive*) if $A\mathbb{R}^n_+ \subseteq \mathbb{R}^n_+$ ($A\mathbb{R}^n_+ \subset \mathrm{Int}(\mathbb{R}^n_+)$). We say that $A \ge B$ if $A - B$ is non-negative ($A > B$ if $A - B$ is positive).

The following two lemmas are needed for the sequel.

Lemma 3.1. *If $A \ge B \ge 0$ and $C \ge D \ge 0$ then $AC \ge BD$.*

Summing inequalities $(A - B)C \ge 0$ and $B(C - D) \ge 0$, we get the proof. The application of Lemma 3.1 and induction yields the following remark.

Remark. $A_n \ldots A_1 \ge B^n$ if $A_1 \ge B \ge 0, \ldots, A_n \ge B \ge 0$.

Lemma 3.2. *(i) If $A > B \ge 0$, then $k(A) > k(B)$. (ii) If $A \ge B \ge 0$, then $k(A) \ge k(B)$.*

Proof. Let us consider the case (i). Here $a > b$ where $a = \|A\|$ and $b = \|B\|$. We have

$$\|A\|_L \le \|A - B\|_L + \|B\|_L.$$

Using (3.1), we get

$$\|A - B\|_L < \|A - B\| = a - b = \|A\| - \|B\|.$$

Finally, we obtain

$$\|A\|_L < \|A\| - \|B\| + \|B\|_L$$

whence $k(A) > k(B)$. The case (ii) is analogous. □

3.2 Stability for some dynamical system

Consider a system of differential equations

$$\dot{x} = f(x, z(t)), x \in \mathbb{R}^n, z(t) \in \Omega. \tag{3.2}$$

Let $x(t,\, t_0,\, g)$ be a unique solution of system (3.2) such that $x(t_0,\, t_0,\, g) = g$. Let $J(x,\, z)$ be the Jacoby matrix of the vector field $f(x,\, z)$

$$J(x, z) = \partial f / \partial x = (\partial f_i / \partial x_j).$$

By definition, put

$$\alpha(z) = -\min_i \min_{x \in X} J_{ii}(x, z),\ \ X \subset \mathbb{R}^n. \tag{3.3}$$

Suppose system (3.2) satisfies the following assumption. The function $\sum_{i=1}^n x_i$ is the first integral of system (3.2), i.e., $\sum_{i=1}^n f_i(x, z) = 0$. It follows that $J(x,\, z)L \subset L$. Further assume that the map $\exp(tJ)$ is Markov when $t \geq 0$, $x \in X$, and $z \in \Omega$.

Suppose X is a convex compact subset in L and X is invariant with respect to dynamical system (3.2). Now suppose there exists a matrix $B \geq 0$ such that the following conditions hold:

(a) $\forall z \in \Omega,\ x \in X\ \ J(x,z) + \alpha(z)I + \zeta \geq B$, where I is the identity matrix and $\zeta \geq 0$ is constant;

(b) there exists a number $n_0 \in \mathbb{N}$ such that $k((I+B)^{n_0}) > 0$.

Theorem 3.3. *The map $x(t_0 + \tau,\, t_0,\, \cdot) : X \to X$ is contracting when $t_0 \in \mathbb{R}$ and $\tau > 0$. If*

$$\sup_{t_0} \exp\left(\int_{t_0}^{t_0+\tau} \alpha(z(t))dt\right) = C(\tau) < \infty,$$

then the contraction coefficient is bounded by $q < 1$ uniformly with respect to t_0.

Proof. Let $\Phi(t_0 + \tau,\, t_0,\, g) = \partial x(t_0 + \tau,\, t_0,\, g)/\partial g$ be the Jacoby matrix of the map $x(t_0 + \tau,\, t_0,\, \cdot) : X \to X$. The map Φ is Markov. Consider

$$\Psi(\tau) = \Phi \exp\left(\int_{t_0}^{t_0+\tau} \alpha(z(t))dt\right)$$

and

$$A(\tau) = J\big(x(t_0+\tau, t_0, g), z(t)\big) + \alpha(z(t))I.$$

Using the inequality $A(\tau) \geq B$, the equation $\dot{\Psi} = A(\tau)\Psi(\tau)$, $\Psi(0) = I$, and lemma 3.1, we get the following estimates

$$\Psi(\tau) \geq \exp(\tau B) \geq \frac{\tau^{n_0}\exp(-\tau)}{(n_0)!}(I+B)^{n_0}.$$

Taking into account lemma 3.2, we obtain

$$k(\Psi(\tau)) \geq \frac{\tau^{n_0}\exp(-\tau)}{(n_0)!} k((I+B)^{n_0}) > 0.$$

Since

$$k(\Phi) = k(\Psi)\exp\left(-\int_{t_0}^{t_0+\tau} \alpha(z(t))dt\right),$$

and the map Φ is Markov, we see that $\sup_{g \in X} ||\Phi||_L < 1$. It can be proved that under the condition $C(\tau) < \infty$ we have a uniform estimate.

Finally, by definition, put

$$\gamma(s) = (1-s)g_1 + sg_2, 0 \le s \le 1$$

where g_1, $g_2 \in X$. Now note that $\gamma'(s) = g_2 - g_1 \in L$ and

$$\|x(t_0+\tau, t_0, g_2) - x(t_0+\tau, t_0, g_1)\| \le \int_0^1 \|\Phi\big(t_0+\tau, t_0, \gamma(s)\big)\gamma'(s)\| ds$$

$$\le \sup_{g\in X}\|\Phi\|_L \int_0^1 \|\gamma'(s)\| ds = \sup_{g\in X}\|\Phi\|_L \|g_2 - g_1\|.$$

This concludes the proof. $\square$

4 Proofs of theorems 2.1–2.4

Now we prove the theorem 2.1.

Proof of theorem 2.1. Consider the Jacoby matrix of vector field (1.3)

$$J(u,\lambda) = \begin{pmatrix} -\beta & \lambda & 0 & \dots & 0 & 0 & (1-u_1) \\ V & -\beta & \lambda & \dots & 0 & 0 & (u_1-u_2) \\ 0 & V & -\beta & \dots & 0 & 0 & (u_2-u_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\beta & \lambda & (u_{m-2}-u_{m-1}) \\ 0 & 0 & 0 & \dots & V & -\beta & (u_{m-1}-u_m) \\ \lambda & 0 & 0 & \dots & 0 & V & -(1-u_m) \end{pmatrix}$$

where $\beta = \lambda + V$.

Let us show that the conditions of theorem 3.3 hold. First note that the set X is the convex compact subset in $\mathbb{R}^{m+1}$.

Now note that the map $\exp(tJ)$ is Markov. We get

$$\alpha(\lambda) = -\min_i \min_{x\in X} J_{ii}(x,\lambda) = \max(\lambda + r, 1 + r/m).$$

Further note that $J(x,\lambda(t)) + \alpha(\lambda(t))I \ge B$ where

$$B = \begin{pmatrix} 0 & \bar\lambda & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \bar\lambda & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \bar\lambda & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \bar\lambda & 0 & 0 & \dots & 0 & 0 & \bar\lambda \end{pmatrix}$$

and $\bar\lambda$ is given by (2.1). The matrix $(I+B)^m$ has the following structure

$$(I+B)^m = \begin{pmatrix} * & * & * & \dots & * & * & 0 \\ 0 & * & * & \dots & * & * & 0 \\ 0 & 0 & * & \dots & * & * & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & * & 0 \\ 0 & 0 & 0 & \dots & 0 & * & 0 \\ * & * & * & \dots & * & * & * \end{pmatrix}$$

where any $*$ denotes a positive number. It follows that $k((I+B)^m) > 0$.

Since $\sum_{i=1}^m u_i + V$ is the first integral of system (1.3), $u(t, \tau, g)$ moves along the plane $\sum_{i=1}^m u_i + V = r$.

Lemma 4.1. *The set* $\operatorname{Int} X$ *is invariant with respect to (1.3).*

Proof. Assume the converse. Then $u(\tau, \tau, g) = (u_1(\tau), \ldots, u_m(\tau), V(\tau))$ satisfies the following inequalities

$$1 = u_0 > u_1 > u_2 > \ldots > u_m > 0,\ V > 0 \tag{4.1}$$

when $g \in \operatorname{Int} X$ and $t \in [\tau, t_0)$, but at the moment t_0 some inequalities in (4.1) change to equalities.

If $V(t_0) = 0$, then $\left.\dfrac{dV}{dt}\right|_{t=t_0-0} = \lambda(t_0-0)u_1(t_0) \geq \lambda(t_0-0)r/m > 0$ and we arrive at contradiction with $V(t_0) - V(t) < 0$ for $t < t_0$. Therefore, we have $V(t_0) > 0$.

Using the virtual variable $u_{m+1} \equiv 0$, we change f_m in (1.3') to

$$f_m(u, \lambda) = \lambda(u_{m+1} - u_m) + V(u_{m-1} - u_m).$$

Since $1 = u_0 > u_{m+1} = 0$, we see that the following two cases are only possible. 1) There exists i such that $u_{i-1}(t_0) = u_i(t_0) > u_{i+1}(t_0)$, 2) There exists j such that $u_{j-1}(t_0) > u_j(t_0) = u_{j+1}(t_0)$.

Let us consider the case 1). Here $\left.\dfrac{du_{i-1}}{dt}\right|_{t=t_0-0} = V(t_0)(u_{i-2}(t_0) - u_{i-1}(t_0)) \geq 0$ and $\left.\dfrac{du_i}{dt}\right|_{t=t_0-0} = \lambda(t_0-0)(u_{i+1}(t_0) - u_i(t_0)) < 0$. We have $\left.\dfrac{d(u_{i-1}-u_i)}{dt}\right|_{t=t_0-0} > 0$ and we arrive at contradiction with $[u_{i-1}(t_0) - u_i(t_0)] - [u_{i-1}(t) - u_i(t)] < 0$ when $t < t_0$. The case 2) is analogous. This concludes the proof. The same idea is used in [2, Lemma 1]. $\square$

We claim that X is invariant with respect to (1.3). Indeed, this follows from lemma 4.1 and continuity of the map $u(t, \tau, \cdot)$.

Finally note that the application of theorem 3.3 yields the conclusions of theorem 2.1. $\square$

Proof of theorem 2.2. Without loss of generality it can be assumed that $\lambda = const$. Therefore $T_N(t, \tau) = T_N(t - \tau)$. Now the theorem can be proved by the standard methods used in [2].

The set $C(X)$ of all continuous functions over X is a Banach space with respect to the norm $\|f\| = \max_{x \in X}|f|$. Let $T(t)$ by a semigroup over C(X) such that

$$T(t)f(g) = f(u(t, 0, g)), \tag{4.2}$$

where $f \in C(X)$. The semigroups $T(t)$ and $T_N(t)$ are strongly continuous. Let A (A_N) be the generator of the semigroup $T(t)$ ($T_N(t)$). By $D(A)$ denote the domain of A. From (4.2) it follows that $f \in D(A)$ if $f \in C(X)$ and

$$\partial f/\partial u_1, \ldots, \partial f/\partial u_m, \partial f/\partial V,\quad \partial^2 f/\partial^2 u_1, \ldots, \partial^2 f/\partial^2 u_m, \partial^2 f/\partial^2 V \in C(X).$$

By D denote the set of all functions f satisfying the last conditions. It is easy to prove that the set D is the core of the operator A. It can be proved (see [2]) that for all $f \in D$

$$\lim_{N\to\infty} \sup_{x \in X_N} |A_N f(x) - Af(x)| = 0.$$

Using the MPs convergence theorems (see, e.g, [4, chapter 1, Theorem 6.1]) we get (2.2). $\square$

Proof of theorem 2.3. The theorem follows from theorem 2.2 and e.g. [4, chapter 4, theorem 2.11]. $\square$

Proof of theorem 2.4. Because of assertion (d) of theorem 2.1 there exists a unique distribution F over X such that d is invariant with respect to dynamical system (1.3). Therefore $F = \varepsilon_{g^*}$. Theorem 2.2 implies that the invariant distributions μ_N converge to ε_{g^*}. The same idea was used in [2], theorem 5. $\square$

5 Remark about model from [2]

The model considered in [2] is a queuing system S_N, with N identical infinite-buffer FCFS single servers, fed by a Poisson arrival flow of rate $N\lambda(t)$ and with i.i.d exponential service times of mean 1. Upon its arrival each task chooses two servers at random (with probability $1/N^2$) and then selects, among the chosen ones, the server with lowest number of tasks in the buffer (including the task in service). If there happens to be more than one server with lowest number queue-size, the task selects one of them randomly. S_N is the system of type $M|M|N|\infty$. We consider a system S_N^m of type $M|M|N|m$. If the task selects a server with m tasks in the queue then the task leaves the system.

Mean field equations for S_N^m have the following form

$$\begin{aligned} \dot{u}_k &= (u_{k+1} - u_k) + \lambda(u_{k-1}^2 - u_k^2), \ k = 1, \dots, m-1, \\ \dot{u}_m &= -u_m + \lambda(u_{m-1}^2 - u_m^2), \end{aligned} \tag{5.1}$$

where $1 \geq u_1 \geq \ldots \geq u_m \geq 0$. By definition, put

$$\dot{V} = u_1 + \lambda u_m^2 - \lambda, V \geq 0, V(0) = m - \sum_{i=1}^{m} u_i(0). \tag{5.1'}$$

Let $x \in \mathbb{R}^{m+1}$ be the vector

$$(x_1, \dots, x_{m+1}) = (u_1, \dots, u_m, V).$$

The following system corresponds to system (5.1), (5.1')

$$\dot{x} = f(x, \lambda) \tag{5.2}$$

Here by definition, put

$$X = \{x \in \mathbb{R}^{m+1} | 1 \geq x_1 \geq \ldots \geq x_m \geq 0, \ x_{m+1} + \sum_{i=1}^{m} x_i = m, x_{m+1} \geq 0\}.$$

¿From [2, Lemma 1] it follows that the set X is invariant with respect to ¿system (5.2). By direct calculation, we get

$$J(x,\lambda) = \begin{pmatrix} -\beta_1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 2\lambda u_1 & -\beta_2 & 1 & \dots & 0 & 0 & 0 \\ 0 & 2\lambda u_2 & -\beta_3 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\beta_{m-1} & 1 & 0 \\ 0 & 0 & 0 & \dots & 2\lambda u_{m-1} & -\beta_m & 0 \\ 1 & 0 & 0 & \dots & 0 & 2\lambda u_m & 0 \end{pmatrix}$$

where $\beta_i = 1 + 2\lambda u_i$. From (3.3) it follows that $\alpha(\lambda) = 1 + 2\lambda$. Clearly, $J(x, \lambda) + \alpha(\lambda)I \geq B$ where

$$B = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}.$$

The matrix $(I+B)^m$ has the following structure

$$(I+B)^m = \begin{pmatrix} * & * & * & \dots & * & * & 0 \\ 0 & * & * & \dots & * & * & 0 \\ 0 & 0 & * & \dots & * & * & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & * & 0 \\ 0 & 0 & 0 & \dots & 0 & * & 0 \\ * & * & * & \dots & * & * & * \end{pmatrix}$$

where any $*$ denotes a positive number. It follows that $k((I+B)^m) > 0$.

Using theorem 3.3, we obtain

Theorem 5.1. *For any $\lambda > 0$ there exists a unique exponentially stable stationary solution of system (5.2).*

Consider the system of differential equations

$$\dot{x} = f(x, \lambda(t)), \tag{5.3}$$

where $f(x, \lambda)$ is defined by (5.2) and the piecewise continuous function $\lambda(t)$ is T-periodic. Similarly, using theorem 3.3, we get

Theorem 5.2. *Suppose $\inf \lambda(t) > 0$ and $\sup \lambda(t) < \infty$; then there exists a unique exponentially stable T-periodic solution of system (5.3).*

Let us remark that though the method of paper [2] yields the proof of the global stability of system (5.2), it cannot prove the convergence of the solution of system (5.3) to a T-periodic orbit.

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